Alternating permutations with restrictions and standard Young tableaux

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Abstract. In this paper, we give bijections between the set of 4123-avoiding down-up alternating permutations of length $2n$ and the set of standard Young tableaux of shape (n, n, n), and between the set of 4123-avoiding down-up alternating permutations of length $2n-1$ and the set of shifted standard Young tableaux of shape $(n+1, n, n-1)$ via an intermediate structure of Yamanouchi words. Moreover, we get the enumeration of 4123-avoiding up-down alternating permutations of even and odd length by presenting bijections between 4123-avoiding up-down alternating permutations and standard Young tableaux.

Key words: alternating permutation, standard Young tableau, shifted standard Young tableau.
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1 Introduction

A permutation $\pi = \pi_1\pi_2\ldots\pi_n$ of length n on $[n] = \{1, 2, \ldots, n\}$ is said to be an *up-down alternating* permutations if $\pi_1 < \pi_2 > \pi_3 < \pi_4 > \cdots$. Similarly, π is said to be a *down-up* alternating permutation if $\pi_1 > \pi_2 < \pi_3 > \pi_4 < \cdots$. We denote by $\mathcal{UD}_n$ and $\mathcal{DU}_n$ the set of up-down and down-up alternating permutations of length n, respectively. Note that the *complement* map $\pi = \pi_1\pi_2\ldots\pi_n \longmapsto (n+1-\pi_1)(n+1-\pi_2)\ldots(n+1-\pi_n)$ is a bijection between the set $\mathcal{UD}_n$ and the set $\mathcal{DU}_n$. Denote by $\mathcal{S}_n$ the set of all permutations on $[n]$. Given a permutation $\pi = \pi_1\pi_2\ldots\pi_n \in \mathcal{S}_n$ and a permutation $\tau = \tau_1\tau_2\ldots\tau_k \in \mathcal{S}_k$, we say that π contains *pattern* τ if there exists a subsequence $\pi_{i_1}\pi_{i_2}\ldots\pi_{i_k}$ of π that is order-isomorphic to τ. Otherwise, π is said to *avoid* the pattern τ or be τ-*avoiding*.

The classic problem of enumerating permutations avoiding a given pattern has received a great deal of attention and has led to interesting variations. For a thorough summary of the current status of research, see Bóna's book [1]. As an interesting variation, Mansour [7] studied alternating permutations avoiding a given pattern. Alternating permutations have the intriguing property [7, 10] that for any pattern of length 3, the number of alternating permutations avoiding that pattern is given by Catalan numbers. This property is shared by the

ordinary permutations. This coincidence suggests that pattern avoidance in alternating permutations and in ordinary permutations may be closely related, which motivates the pattern avoidance in alternating permutations. Guibert and Linusson [4] showed that doubly alternating Baxter permutations are counted by the Catalan numbers and Ouchterlony [8] studied the problem of enumerating doubly alternating permutations avoiding patterns of length 3 and 4. Recently, Lewis [5] initiated the study of enumerating alternating permutations avoiding a given pattern of length 4. Let $\mathcal{UD}_n(\tau)$ and $\mathcal{DU}_n(\tau)$ be the set of τ-avoiding up-down and down-up alternating permutations of length n, respectively. Lewis [5] provided bijections between the set $\mathcal{UD}_{2n}(1234)$ and standard Young tableaux of shape (n, n, n), and between the set $\mathcal{UD}_{2n+1}(1234)$ and standard Young tableaux of shape $(n+1, n, n-1)$. By applying hook length formula for standard Young tableaux [9], the number of 1234-avoiding up-down alternating permutations of length $2n$ is given by $\frac{2(3n)!}{n!(n+1)!(n+2)!}$, and the number of 1234-avoiding up-down alternating permutations of length $2n+1$ is given by $\frac{16(3n)!}{(n-1)!(n+1)!(n+3)!}$. Using the method of generating trees, Lewis [6] constructed recursive bijections between the set $\mathcal{UD}_{2n}(2143)$ and the set of standard Young tableaux of shape (n, n, n), and between the set $\mathcal{UD}_{2n+1}(2143)$ and the set of shifted standard Young tableaux of shape $(n+2, n+1, n)$. Using computer simulations, Lewis [6] came up with several conjectures that indicated there are surprising connections between alternating permutations and ordinary permutations.

In this paper, we are concerned with the enumeration of 4123-avoiding down-up and up-down alternating permutations of even and odd length. We establish recursive bijections between the set $\mathcal{DU}_{2n}(4123)$ and the set of standard Young tableaux of shape (n, n, n), and between the set $\mathcal{DU}_{2n-1}(4123)$ and the set of shifted standard Young tableaux of shape $(n+1, n, n-1)$ via an intermediate structure of Yamanouchi words. Consequently, we prove the conjectures, posed by Lewis [6], that $|\mathcal{UD}_{2n}(1432)| = |\mathcal{UD}_{2n}(1234)|$ and $|\mathcal{UD}_{2n+1}(1432)| = |\mathcal{UD}_{2n+1}(2143)|$ in the sense that $|\mathcal{UD}_n(1432)| = |\mathcal{DU}_n(4123)|$ by the operation of complement.

Applying the bijections between 4123-avoiding down-up alternating permutations and standard Young tableaux, we show that for $n \geq 1$, 4123-avoiding up-down alternating permutations of $2n+1$ are in one-to-one correspondence with standard Young tableaux of shape $(n+1, n, n-1)$. Moreover, for $n \geq 2$, 4123-avoiding up-down alternating permutations of length $2n$ are in bijection with shifted standard Young tableaux of shape $(n+2, n, n-2)$. As a result, we deduce that $|\mathcal{UD}_{2n}(4123)| = |\mathcal{UD}_{2n}(1234)|$ and $|\mathcal{UD}_{2n+1}(4123)| = |\mathcal{UD}_{2n+1}(1234)|$, as conjectured by Lewis [6].

The paper is organized as follows. In Section 2, we introduce the bijections

between the set $\mathcal{DU}_{2n}(4123)$ and the set of standard Young tableaux of shape (n, n, n), and between the set $\mathcal{DU}_{2n-1}(4123)$ and the set of shifted standard Young tableaux of shape $(n+1, n, n-1)$. In section 3, we get the enumeration of 4123-avoiding up-down alternating permutations of even and odd length.

2 4123-avoiding down-up alternating permutations

In this section, we aim to construct recursive bijections between the set $\mathcal{DU}_{2n}(4123)$ and the set of standard Young tableaux of shape (n, n, n), and between the set $\mathcal{DU}_{2n-1}(4123)$ and the set of shifted standard Young tableaux of shape $(n+1, n, n-1)$. To do this, we need to introduce the following definitions and notations.

A *partition* λ of a positive integer n is defined to be a sequence $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ of nonnegative integers such that $\lambda_1 + \lambda_2 + \ldots \lambda_m = n$ and $\lambda_1 \geq \lambda_2 \ldots \geq \lambda_m$. Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$, the (ordinary) Young diagram of shape λ is the left-justified array of $\lambda_1 + \lambda_2 + \ldots + \lambda_m$ boxes with λ_1 boxes in the first row, λ_2 boxes in the second row, and so on. If λ is a partition with distinct parts then the *shifted* Young diagram of shape λ is an array of cells with m rows, each row indented by one cell to the right with respect to the previous row, and λ_i cells in row i. We shall frequently use the same symbols for things which may have an ordinary or shifted interpretation. It will always be clear which interpretation is meant.

If λ is a Young diagram with n boxes, a *standard Young tableau* of shape λ is a filling of the boxes of λ with $[n]$ so that each element appears in exactly one box and entries increase along rows and columns. We identify boxes in Young diagrams and tableaux using matrix coordinates. For example, the box in the first row and second column is numbered $(1, 2)$.

Given a standard Young tableau T with n entries, we associate T with a word $\chi(T)$ of length n on the alphabet $\{1, 2, 3, \ldots\}$, where $\chi(T)$ is obtained from T by letting the jth letter be the row index of the entry of T containing the number j. The words $\chi(T)$ are known as *Yamanouchi* words [3]. On the other hand, given a Yamanouchi word w, it is straightforward to recover the corresponding tableaux $\chi^{-1}(w)$ by letting the i-th row contain the indices of letters of w that are equal to i. For example, the associated standard Young tableau of the Yamanouchi word 112311223 is illustrated as follows:

1	2	5	6
3	7	8	
4	9		

Given a word w on the alphabet $\{1, 2, \ldots\}$, we define c_i to be the number of entries of w that are equal to i and the *type* of the word w to be the sequence $(c_1, c_2, c_3, \ldots)$. Let $w = w_1 w_2 \ldots w_n$ be a word on the alphabet $\{1, 2, 3\}$. The subsequence $w_1 w_2 \ldots w_j$ is said to be an *initial run* of length j in w if w_{j+1} is the leftmost entry of w that is equal to 3. Similarly, we can define the *final run* of length j to be a subsequence $w_{n+1-j} w_{n+2-j} \ldots w_n$ such that w_{n-j} is the rightmost entry equal to 1. Denote by $\alpha(w)$ and $\beta(w)$ the length of the initial run and the final run of w, respectively. For instance, let $w = 121211231323233$. We have $\alpha(w) = 7$ and $\beta(w) = 6$.

In order to establish the bijections between 4123-avoiding down-up alternating permutations and standard Young tableaux, we consider the following two sets. Given a permutation $\pi = \pi_1 \pi_2 \ldots \pi_n$, let

$$\mathcal{A}(\pi) = \{0\} \cup \{k | \ \exists i < j \ \text{ s.t. } \pi_i = k, \pi_j = k+1 \, \text{and } k \leq \pi_1 - 2\}.$$

Given a word $w = w_1 w_2 \ldots w_n$ on the alphabet $\{1, 2, 3\}$, we define

$$\mathcal{B}(w) = \{0\} \cup \{k | \ w_k w_{k+1} = 12 \text{ and } k \leq \alpha(w) - 2\}.$$

Example 2.1 *Consider* $\pi = 658397(10)142$ *and* $w = 121211231323233$. *We have* $\mathcal{A}(\pi) = \{0, 1, 3\}$ *and* $\mathcal{B}(w) = \{0, 1, 3\}$.

Given a permutation $\pi \in \mathcal{S}_n$ and an element $a \in [n+1]$, there is a unique permutation $\pi' = \pi'_1 \pi'_2 \ldots \pi'_{n+1} \in \mathcal{S}_{n+1}$ such that $\pi'_1 = a$ and the word $\pi'_2 \pi'_3 \ldots \pi'_{n+1}$ is order-isomorphic to π. We denote this permutation by $a \to \pi$. Let $a, b \in [n+2]$ with $b < a$. Denote by $(a, b) \to \pi$ the permutation $u = u_1 u_2 \ldots u_{n+2}$ such that $u_1 = a$, $u_2 = b$ and $u_3 u_4 \ldots u_{n+2}$ is order-isomorphic to π. More precisely, the permutation u is defined by

$$u_i = \begin{cases} a, & i = 1, \\ b, & i = 2, \\ \pi_{i-2}, & \pi_{i-2} < b, \\ \pi_{i-2} + 1, & b \leq \pi_{i-2} < a - 1, \\ \pi_{i-2} + 2, & \pi_{i-2} \geq a - 1. \end{cases}$$

We start with two lemmas that will be essential in the construction of the bijections between 4123-avoiding down-up alternating permutations and standard Young tableaux. First, we present the following simple observation that will be of use in the subsequent proofs of Lemmas.

Observation 2.2 *Let $\pi = \pi_1\pi_2\ldots\pi_n \in \mathcal{S}_n$ with $\mathcal{A}(\pi) = \{a_0, a_1, \ldots, a_p\}$, where $p \geq 0$ and $0 = a_0 < a_1 < a_2 \ldots < a_p$. Assume that $a_{p+1} = \pi_1$. For any integers r and s with $a_j < r < s \leq a_{j+1}$, suppose that $\pi_l = r$ and $\pi_m = s$. Then we have $l > m$.*

Lemma 2.3 *Let $\pi = \pi_1\pi_2\ldots\pi_n \in \mathcal{DU}_n(4123)$ with $\mathcal{A}(\pi) = \{a_0, a_1, \ldots, a_p\}$, where $p \geq 0$ and $0 = a_0 < a_1 < a_2 \ldots < a_p$. Assume that $a_{p+1} = \pi_1$. If $\pi' = (a, b) \to \pi$ is a permutation in $\mathcal{DU}_{n+2}(4123)$, then $b \leq \pi_1$ and there exists an integer j such that $a_{j+1} + 2 \geq a > b \geq a_j + 1$.*

Proof. Let $\pi' = \pi'_1\pi'_2\ldots\pi'_{n+2}$. Recall that $\pi'_1 = a$ and $\pi'_2 = b$. Since π is a down-up alternating permutation, we have $b \leq \pi_1 = a_{p+1}$. Suppose that $a_{j+1} \geq b \geq a_j + 1$ for some integer j with $0 \leq j \leq p$. We claim that $a \leq a_{j+1} + 2$. Otherwise, assume that $a > a_{j+1}+2$. Then we have two cases. If $j = p$, then the subsequence $ab(a_{p+1}+1)(a_{p+1}+2)$ is order-isomorphic to 4123 in π' since $\pi'_3 = \pi_1+1 = a_{p+1}+1$ and $\pi'_2 = b < \pi'_3$. If $j < p$, then according to the definition of $\mathcal{A}(\pi)$, there exists integers l and m with $l < m$ such that $\pi_l = a_{j+1}$ and $\pi_m = a_{j+1} + 1$. Note that $\pi'_{l+2} = \pi_l + 1 = a_{j+1} + 1$ and $\pi'_{m+2} = \pi_m + 1 = a_{j+1} + 2$. Then the subsequence $\pi'_1\pi'_2\pi'_{l+2}\pi'_{m+2}$ forms a 4123 pattern in π'. Hence, we deduce that $a_j + 1 \leq b < a \leq a_{j+1} + 2$. This completes the proof. ∎

Lemma 2.4 *Let $\pi = \pi_1\pi_2\ldots\pi_n \in \mathcal{DU}_n(4123)$ with $\mathcal{A}(\pi) = \{a_0, a_1, \ldots, a_p\}$, where $p \geq 0$ and $0 = a_0 < a_1 < a_2 \ldots < a_p$. Assume that $a_{p+1} = \pi_1$. Let a, b be two integers such that $a_{j+1}+2 \geq a > b \geq a_j+1$ and $b \leq \pi_1$. Then $\pi' = (a, b) \to \pi$ is a permutation in $\mathcal{DU}_{n+2}(4123)$ satisfying that*

(i) *if $b = a_j+1$ and $j \geq 1$, we have $\mathcal{A}(\pi') = \{a_0, a_1, \ldots, a_{j-1}, b\}$ when $a > b+1$ and $\mathcal{A}(\pi') = \{a_0, a_1, \ldots, a_{j-1}\}$ when $a = b + 1$;*

(ii) *otherwise, we have $\mathcal{A}(\pi') = \{a_0, a_1, \ldots, a_j, b\}$ when $a > b + 1$ and $\mathcal{A}(\pi') = \{a_0, a_1, \ldots, a_j\}$ when $a = b + 1$;*

Proof. Since $b \leq \pi_1$, the permutation π' is a down-up alternating permutation. Now we proceed to show that π' avoids the pattern 4123. Let $\pi' = \pi'_1\pi'_2\ldots\pi'_{n+2}$.

Suppose that there is a subsequence $\pi'_k\pi'_l\pi'_m\pi'_q$ with $k < l < m < q$ which is order-isomorphic to 4123. Since the subsequence $\pi'_3 \dots \pi'_{n+2}$ is order-isomorphic to π, we have either $k = 1$ or $k = 2$. If $k = 2$, since $\pi'_2 < \pi'_3$, the subsequence $\pi'_3\pi'_l\pi'_m\pi'_q$ is order-isomorphic to the pattern 4123. If $k = 1$ and $l > 2$, then the subsequence $\pi'_3\pi'_l\pi'_m\pi'_q$ is an instance of 4123 since $\pi'_3 \geq \pi_1 + 1$ and $\pi'_1 = a \leq \pi_1 + 2$. Thus, it follows that $k = 1$ and $l = 2$. Recall that $\pi'_1 = a$, $\pi'_2 = b$, which implies that $b < \pi'_m < \pi'_q < a$. From this, we deduce that $a_j \leq b - 1 < \pi'_m - 1 < \pi'_q - 1 < a - 1 \leq a_{j+1} + 1$. Note that $\pi'_m = \pi_{m-2} + 1$ and $\pi'_q = \pi_{q-2} + 1$. So we have $a_j < \pi_{m-2} < \pi_{q-2} \leq a_{j+1}$. This contradicts with Observation 2.2. Thus, the permutation π' is in $\mathcal{DU}_{n+2}(4123)$.

It remains to prove that the permutation π' verifies the points (i) and (ii). It is easily seen for any $0 \leq k \leq j - 1$, we have $a_k \in \mathcal{A}(\pi')$. Now we proceed to to show that there exists no integer k such that $k > b$ and $k \in \mathcal{A}(\pi')$. Otherwise, suppose that k is such an integer. According to the definition of $\mathcal{A}(\pi')$, there exists integers l and m with $l < m$ such that $\pi'_l = k$, $\pi'_m = k + 1$ and $k \leq \pi'_1 - 2 = a - 2 \leq a_{j+1}$. This implies that $\pi_{l-2} = k - 1$ and $\pi_{m-2} = k$. So we have $a_j \leq b - 1 < k - 1 = \pi_{l-2} < \pi_{m-2} = k \leq a_{j+1}$. This contradicts with Observation 2.2. So we conclude that there exists no integer k such that $k > b$ and $k \in \mathcal{A}(\pi')$.

If $a > b + 1$, then we have $b \in \mathcal{A}(\pi')$ since $b + 1$ appears right to b in π' and $b \leq a - 2 = \pi'_1 - 2$. If $a = b + 1$, then $b \notin \mathcal{A}(\pi')$ since $b + 1$ appears left to b in π'. Moreover, when $b = a_j + 1$ and $j \geq 1$, we have $a_j \notin \mathcal{A}(\pi')$ since $a_j + 1$ appears left to a_j in π'. Otherwise, we have $a_j \in \mathcal{A}(\pi')$. Hence (i) and (ii) are verified. This completes the proof. ∎

Now we proceed to construct a recursive bijection between the set $\mathcal{DU}_{2n}(4123)$ and the set of Yamanouchi words on the alphabet $\{1, 2, 3\}$ of type (n, n, n).

Theorem 2.5 *There is a bijection ϕ between the set $\mathcal{DU}_{2n}(4123)$ and the set of Yamanouchi words on the alphabet $\{1, 2, 3\}$ of type (n, n, n) satisfying that $\pi_1 = \alpha(\phi(\pi))$ and $\mathcal{A}(\pi) = \mathcal{B}(\phi(\pi))$ for any permutation $\pi = \pi_1\pi_2 \dots \pi_{2n} \in \mathcal{DU}_{2n}(4123)$.*

Proof. Now, we define a map ϕ from $\mathcal{DU}_{2n}(4123)$ to the set of Yamanouchi words on the alphabet $\{1, 2, 3\}$ of type (n, n, n) in terms of a recursive procedure. For $n = 1$, we define $\phi(21) = 123$. It is clear that for $n = 1$, the claim holds. Now, given any permutation $\pi = \pi_1\pi_2 \dots \pi_{2n+2} \in \mathcal{DU}_{2n+2}(4123)$ for $n \geq 1$, we proceed to construct a Yamanouchi word $w = \phi(\pi)$. Let $\pi' = \pi'_1\pi'_2 \dots \pi'_{2n}$ be a 4123-avoiding down-up alternating permutation and $\pi = (a, b) \rightarrow \pi'$ where $a = \pi_1$ and $b = \pi_2$. Let $\mathcal{A}(\pi') = \{a_0, a_1, \dots, a_p\}$ with $p \geq 0$ and $0 = a_0 < a_1 < a_2 < \dots < a_p$. Assume that $v = \phi(\pi') = v_1v_2 \dots v_{3n}$. By the induction hypothesis, v is a Yamanouchi word on the alphabet $\{1, 2, 3\}$ of type (n, n, n) with the property

that $\alpha(v) = \pi'_1$ and $\mathcal{B}(v) = \mathcal{A}(\pi')$. Assume that $a_{p+1} = \alpha(v) = \pi'_1$. By Lemma 2.3, we have $a_{j+1} + 2 \geq a > b \geq a_j + 1$ for some integer j and $b \leq \pi'_1 = \alpha(v)$. Now we proceed to construct a word $w = \phi(\pi)$ on the alphabet $\{1, 2, 3\}$ from v by distinguishing the following two cases.

(i) If $a = b + 1$, then set $w = v_1 v_2 \ldots v_{b-1} \mathbf{123} v_b \ldots v_{3n}$.

(ii) If $a > b + 1$, then set

$$w = v_1 v_2 \ldots v_{b-1} \mathbf{12} v_b \ldots v_{a-2} \mathbf{3} v_{a-1} \ldots v_{3n}.$$

Clearly, the obtained word w is a Yamanouchi word of type $(n+1, n+1, n+1)$ with an initial run of length a, that is, $\alpha(w) = a$. It remains to show that $\mathcal{A}(\pi) = \mathcal{B}(w)$.

For the case $a = b + 1$, it is easy to check that when $b = a_j + 1$ and $j \geq 1$ $\mathcal{B}(w) = \{a_0, a_1, \ldots, a_{j-1}\}$. Otherwise, we have $\mathcal{B}(w) = \{a_0, a_1, \ldots, a_j\}$. By Lemma 2.4, we deduce that $\mathcal{B}(w) = \mathcal{A}(\pi)$.

For the case $a > b + 1$, suppose that $w = w_1 w_2 \ldots w_{3n+3}$. Since $w_b w_{b+1} = 12$ and $b \leq a - 2 = \alpha(w) - 2$, we have $b \in \mathcal{B}(w)$. Moreover, if $b \geq 2$, we have $b - 1 \notin \mathcal{B}(w)$ since $w_b = 1$. It remains to show that there exists no integer k with $k > b$ such that $k \in \mathcal{B}(w)$. Otherwise, assume that there is such an integer k. According to the definition of $\mathcal{B}(w)$, we have $w_k w_{k+1} = 12$ and $k \leq \alpha(w) - 2 = a - 2$. Note that $w_{b+1} = 2$, which implies that $k \geq b + 2$. So, we have $w_k w_{k+1} = v_{k-2} v_{k-1} = 12$ with $k \leq a - 2 \leq a_{j+1}$. This implies that $k - 2 \in \mathcal{B}(v)$. However we have $a_j + 1 \leq b \leq k - 2 \leq a_{j+1} - 2$. This contradicts with the definition of $\mathcal{B}(v)$. Thus we deduce that $\mathcal{B}(w) = \{a_0, a_1, \ldots, a_{j-1}, b\}$ when $b = a_j + 1$ and $j \geq 1$. Otherwise, we have $\mathcal{B}(w) = \{a_0, a_1, \ldots, a_j, b\}$. By Lemma 2.4, we deduce that $\mathcal{B}(w) = \mathcal{A}(\pi)$.

We conclude that the the obtained word w is a Yamanouchi word of type $(n+1, n+1, n+1)$ such that $\alpha(w) = a = \pi_1$ and $\mathcal{A}(\pi) = \mathcal{B}(w)$.

It is sufficient to construct the inverse mapping of ϕ to show that this is a bijection. Given a Yamanouchi word $w = w_1 w_2 \ldots w_{3n}$ of type (n, n, n) with $\mathcal{B}(w) = \{a_0, a_1, \ldots, a_p\}$ with $p \geq 0$ and $0 = a_0 < a_1 < \ldots < a_p$, we wish to recover a 4123-avoiding down-up alternating permutations $\phi^{-1}(w)$ in terms of a recursive procedure. Assume that $a_{p+1} = \alpha(w)$. If $w = 123$, then define $\phi^{-1}(w) = 21$. Obviously, we have $\alpha(w) = 2$ and $\mathcal{A}(\phi^{-1}(w)) = \mathcal{B}(w) = \{0\}$. Clearly, the claim holds for $n = 1$. For $n \geq 2$, set $a = \alpha(w)$. Now we proceed to associate w with an ordered pair (v, b) by the following procedure.

(a) If $w_{a+2} = 3$, then let $b = a_p$ and v be a word obtained from w by removing w_b, w_{b+1} and w_{a+1} from w.

(*b*) If $w_{a+2} \neq 3$, then find the largest integer q such that $q \leq a-1$ and $w_q w_{q+1} = 12$. Let $b = q$ and v be a word obtained from w by removing w_b, w_{b+1} and w_{a+1} from w.

Finally, we define $\phi^{-1}(w) = (a, b) \to \phi^{-1}(v)$.

Now we proceed to prove that the map ϕ^{-1} is the desired map. For the case $w_{a+2} = 3$, since w is a Yamanouchi word with $w_{a+1} = 3$ and $w_{a+2} = 3$, there are at least two occurrences of 2's left to w_{a+1} and the first occurrence of 2 is preceded immediately by an entry 1. This guarantees that there exists at least one subsequence $w_k w_{k+1} = 12$ with $k \leq a-2$, that is, $\mathcal{B}(w) \neq \{0\}$. Hence we have $b = a_p > 0$. For the case $w_{a+2} \neq 3$, the property of the Yamanouchi word ensures that there exists at least one subsequence $w_k w_{k+1} = 12$ with $k \leq a-1$. Thus, in either case, the word v is a Yamanouchi word of type $(n-1, n-1, n-1)$.

Suppose that $\mathcal{B}(v) = \{c_0, c_1, \ldots, c_m\}$ with $m \geq 0$ and $0 = c_0 < c_1 < \ldots < c_m$. Assume that $c_{m+1} = \alpha(v)$. Now we proceed to show that the obtained permutation is in $\mathcal{DU}_{2n}(4123)$ satisfying that $\mathcal{A}(\phi^{-1}(w)) = \mathcal{B}(w)$ and the first element of $\phi^{-1}(w)$ is equal to $a = \alpha(w)$ by considering the following cases.

- If $w_{a+2} = 3$, then we have $\alpha(v) = a-2$ since v has an initial run of length $a-2$. In this case, we have $a = \alpha(v) + 2 = c_{m+1} + 2$. Moreover, since $b = a_p$, there is no subsequence $w_k w_{k+1} = 12$ in the subsequence $w_{b+2} \ldots w_a$ with $k \leq \alpha(w) - 2 = a - 2$. So, we have $c_m \leq b - 1$. By the induction hypothesis, the permutation $\phi^{-1}(v)$ is in $\mathcal{DU}_{2n-2}(4123)$ whose first element equals $\alpha(v)$ and $\mathcal{A}(\phi^{-1}(v)) = \mathcal{B}(v) = \{c_0, c_1, \ldots, c_m\}$. By Lemma 2.4, we have $\phi^{-1}(w) = (a, b) \to \phi^{-1}(v)$ is in $\mathcal{DU}_{2n}(4123)$ since $c_{m+1} + 2 = a > b \geq c_m + 1$ and $b = a_p \leq a - 2 = \alpha(v)$. Observe that $w_b = 1$. This ensures that $w_{b-1} w_b \neq 12$ and $b-1 \notin \mathcal{B}(w)$ when $b \geq 2$. Thus, we derive that if $c_m = b-1$ and $m \geq 1$ then we have $\mathcal{B}(v) = \{a_0, a_1, \ldots, a_{p-1}, c_m\}$. Otherwise, we have $\mathcal{B}(v) = \{a_0, a_1, \ldots, a_{p-1}\}$. Since $b = a_p \leq \alpha(w) - 2 = a - 2 < a - 1$, we can verify that $\mathcal{A}(\phi^{-1}(w)) = \mathcal{B}(w)$ by Lemma 2.4.

- If $w_{a+2} \neq 3$, then we have $\alpha(v) \geq a-1$ since v has an initial run of length at least $a-1$. Since $b \leq a-1$, we have $b \leq \alpha(v) = c_{m+1}$. This implies that there exists an entry j such that $c_{j+1} \geq b \geq c_j + 1$. Since there is no subsequence of $w_k w_{k+1} = 12$ in the subsequence $w_{b+2} \ldots w_a$, we have $a \leq c_{j+1} + 2$. By the induction hypothesis, the permutation $\phi^{-1}(v)$ is in $\mathcal{DU}_{2n-2}(4123)$ whose first element equals $\alpha(v)$ and $\mathcal{A}(\phi^{-1}(v)) = \mathcal{B}(v) = \{c_0, c_1, \ldots, c_m\}$. Thus, by Lemma 2.4, it follows that $\phi^{-1}(w) = (a, b) \to \phi^{-1}(v)$ is in $\mathcal{DU}_{2n}(4123)$ since $c_{j+1} + 2 \geq a > b \geq c_j + 1$ and $b \leq c_{j+1} \leq c_{m+1} = \alpha(v)$. Note that $w_b = 1$. It follows that $w_{b-1} w_b \neq 12$ and $b - 1 \neq \mathcal{B}(w)$ when $b \geq 2$.

– If $b = a_p$, then for each k with $0 \le k \le p-1$, we have $a_k \in \mathcal{B}(v)$. Thus, we have $\mathcal{B}(v) = \{a_0, a_1, \ldots, a_{p-1}, c_j, c_{j+1}, \ldots, c_m\}$ when $b = c_j + 1$ and $j \ge 1$. Otherwise, we have $\mathcal{B}(v) = \{a_0, a_1, \ldots, a_{p-1}, c_{j+1}, \ldots, c_m\}$. In this case, since $b = a_p \le \alpha(w) - 2 = a - 2 < a - 1$, by Lemma 2.4, we have $\mathcal{A}(\phi^{-1}(w)) = \{a_0, \ldots, a_{p-1}, a_p\} = \mathcal{B}(w)$.

– If $b \ne a_p$, then for each k with $0 \le k \le p$, we have $a_k \in \mathcal{B}(v)$. Thus, we have $\mathcal{B}(v) = \{a_0, a_1, \ldots, a_p, c_j, c_{j+1}, \ldots, c_m\}$ when $b = c_j + 1$ and $j \ge 1$. Otherwise, $\mathcal{B}(v) = \{a_0, a_1, \ldots, a_p, c_{j+1}, \ldots, c_m\}$. In this case, since $b \ne a_p$, we have $b = a - 1$. By Lemma 2.4, it is easily seen that $\mathcal{A}(\phi^{-1}(w)) = \{a_0, a_1, \ldots, a_p\} = \mathcal{B}(w)$.

Hence, the map ϕ is a desired bijection. This completes the proof. ∎

Example 2.6 *Consider a* 4123*-avoiding down-up alternating permutation* $\pi = 63758142$. *We can obtain a Yamanouchi word* w *from* π *recursively as follows:*

$$\begin{array}{lclclcl} \pi = \mathbf{63}758142 & \rightarrow & \mathbf{54}6132 & \rightarrow & \mathbf{41}32 & \rightarrow & 21 \\ w = 12\mathbf{12}11\mathbf{3}23233 & \leftarrow & 121\mathbf{123}233 & \leftarrow & \mathbf{12}1233 & \leftarrow & 123, \end{array}$$

It is easy to check that the word w *has an initial run of length* 6. *Moreover, we have* $\mathcal{A}(\pi) = \{0, 1, 3\}$ *and* $\mathcal{B}(w) = \{0, 1, 3\}$. *Conversely, given a Yamanouchi word* w*, we can recover the* 4123*-avoiding down-up alternating permutation* $\pi = 63758142$ *by reversing the above procedure.*

For $n \ge 1$, let $w = w_1 w_2 \ldots w_{3n}$ be a word on the alphabet $\{1, 2, 3\}$ of type $(n-1, n, n+1)$. Let $\{a_1, a_2, \ldots, a_{n-1}\}$, $\{b_1, b_2, \ldots, b_n\}$ and $\{c_1, c_2, \ldots, c_{n+1}\}$ be the set of indices of letters of w that are equal to 1, 2 and 3, respectively. Suppose that $a_1 < a_2 < \ldots < a_{n-1}$, $b_1 < b_2 < \ldots < b_n$ and $c_1 < c_2 < \ldots < c_{n+1}$. If $b_n < c_n$ and for all $1 \le j \le n-1$, we have $a_j < b_j < c_j$, then the word w is called a *skew Yamanouchi word* of type $(n-1, n, n+1)$. For example, let $w = 112123231323233$ of type $(4, 5, 6)$. We have $\{a_1, a_2, a_3, a_4\} = \{1, 2, 4, 9\}$, $\{b_1, b_2, b_3, b_4, b_5\} = \{3, 5, 7, 11, 13\}$ and $\{c_1, c_2, c_3, c_4, c_5, c_6\} = \{6, 8, 10, 12, 14, 15\}$. Hence, the word w is a skew Yamanouchi word of type $(4, 5, 6)$.

We seek to enumerate $\mathcal{DU}_{2n-1}$ by aping our bijection ϕ for even length permutations. Given a permutation $\pi = \pi_1\pi_2\ldots\pi_{2n-1} \in \mathcal{DU}_{2n-1}(4123)$, we proceed to construct a shifted Yamanouchi word $\psi(\pi)$ on the alphabet $\{1, 2, 3\}$ of type $(n-1, n, n+1)$. If $n = 1$, then define $\psi(1) = 233$. Since the word 233 has an initial run of length 1, we have $\alpha(233) = 1$. Moreover, we have $\mathcal{A}(1) = \mathcal{B}(233) = \{0\}$. For $n \ge 2$, set $\psi(\pi) = \phi(\pi)$. It is easy to check that the arguments in the proof of Theorem 2.5 hold for 4123-avoiding down-up alternating permutation of odd length. As a consequence, we have the following result.

Theorem 2.7 *For $n \geq 1$, the map ψ is a bijection between the set $\mathcal{DU}_{2n-1}(4123)$ and the set of skew Yamanouchi words on the alphabet $\{1,2,3\}$ of type $(n-1,n,n+1)$ satisfying that that $\pi_1 = \alpha(\psi(\pi))$ and $\mathcal{A}(\pi) = \mathcal{B}(\psi(\pi))$ for any permutation $\pi = \pi_1\pi_2\ldots\pi_{2n-1} \in \mathcal{DU}_{2n-1}(4123)$.*

For $n \geq 1$, let $w = w_1w_2\ldots w_{3n}$ be a word on the alphabet $\{1,2,3\}$ of type $(n+1,n,n-1)$. Let $\{a_1,a_2,\ldots,a_{n+1}\}$, $\{b_1,b_2,\ldots,b_n\}$ and $\{c_1,c_2,\ldots,c_{n-1}\}$ be the sets of indices of letters of w that are equal to 1, 2 and 3, respectively. Suppose that $a_1 < a_2 < \ldots < a_{n+1}$, $b_1 < b_2 < b_3 \ldots < b_n$ and $c_1 < c_2 < \ldots < c_{n-1}$. If $a_2 < b_1$ and for all $1 \leq j \leq n-1$, we have $a_{j+2} < b_{j+1} < c_j$, then the word w is called a *shifted Yamanouchi word* of type $(n+1,n,n-1)$. Note that for any shifted standard Young tableau T of shape $(n+1,n,n-1)$, the word $\chi(T)$ is a shifted Yamanouchi word of type $(n+1,n,n-1)$. More precisely, shifted standard Young tableaux of shape $(n+1,n,n-1)$ are in bijection with shifted Yamanouchi words of type $(n+1,n,n-1)$.

Let $w = w_1w_2\ldots w_{3n}$ be a word on the alphabet $\{1,2,3\}$. Denote by $rc(w) = (4-w_n)(4-w_{n-1})\ldots(4-w_1)$ the operation of *reversed complement* of w. let $w = w_1w_2\ldots w_{3n}$ be a skew Yamanouchi word on the alphabet $\{1,2,3\}$ of type $(n-1,n,n+1)$. Let $\{a_1,a_2,\ldots,a_{n-1}\}$, $\{b_1,b_2,\ldots,b_n\}$ and $\{c_1,c_2,\ldots,c_{n+1}\}$ be the set of indices of letters of w that are equal to 1, 2 and 3, respectively. Obviously, the word $rc(w)$ is a word on the alphabet $\{1,2,3\}$ of type $(n+1,n,n-1)$. For $1 \leq i \leq n+1$, set $a'_i = 3n+1-c_{n+2-i}$. For $1 \leq i \leq n$, set $b'_i = 3n+1-b_{n+1-i}$. Similarly, for $1 \leq i \leq n-1$, set $c'_i = 3n+1-a_{n-i}$. According to the definition of the reversed complement of w, the sets $\{a'_1,a'_2,\ldots,a'_{n+1}\}$, $\{b'_1,b'_2,\ldots,b'_n\}$ and $\{c'_1,c'_2,\ldots,c'_{n-1}\}$ are the sets of indices of letters of $rc(w)$ that are equal to 1, 2 and 3, respectively. It is easy to check that $rc(w)$ is a shifted Yamanouchi word of type $(n+1,n,n-1)$. Indeed, the operation of reversed complement turns out to be a bijection between skew Yamanouchi words of type $(n-1,n,n+1)$ and shifted Yamanouchi words of type $(n+1,n,n-1)$. Similarly, the operation of reversed complement is an involution on the set of Yamanouchi words of type (n,n,n). Moreover, the operation of reversed complement transforms an initial run of a word to a final run. Observe that given any ordinary or shifted standard Young tableau T of shape (a,b,c) with the $(1,a)$-entry equal to k, its corresponding ordinary or shifted Yamanouchi word $\chi(T)$ has a final run of length $a+b+c-k$. As an immediate consequence of Theorems 2.5 and 2.7, we have the following results.

Theorem 2.8 *The map $\bar{\phi} = \chi^{-1}\circ(rc)\circ\phi$ is a bijection between the set $DU_{2n}(4123)$ and the set of standard Young tableaux of shape (n,n,n) such that the $(1,n)$-entry of the corresponding tableaux is equal to $3n-\pi_1$ for any permutation $\pi = \pi_1\pi_2\ldots\pi_{2n} \in \mathcal{DU}_{2n}(4123)$.*

Theorem 2.9 *The map $\bar{\psi} = \chi^{-1} \circ (rc) \circ \psi$ is a bijection between the set $DU_{2n-1}(4123)$ and the set of shifted standard Young tableaux of shape $(n+1, n, n-1)$ such that the $(1, n+1)$-entry of the corresponding tableaux is equal to $3n - \pi_1$ for any permutation $\pi = \pi_1\pi_2 \dots \pi_{2n} \in \mathcal{DU}_{2n-1}(4123)$.*

Recall that there are bijections between the set $\mathcal{UD}_{2n}(1234)$ and the standard Young tableaux of shape (n, n, n), and between the set $\mathcal{UD}_{2n+1}(2143)$ and shifted standard Young tableaux of shape $(n+2, n+1, n)$. By the operation of complement, the set $\mathcal{DU}_n(4123)$ are in bijection with the set $\mathcal{UD}_n(1432)$. Thus, from Theorems 2.8 and 2.9, we derive that $|\mathcal{UD}_{2n}(1432)| = |\mathcal{UD}_{2n}(1234)|$ and $|\mathcal{UD}_{2n+1}(1432)| = |\mathcal{UD}_{2n+1}(2143)|$.

3 4123-avoiding up-down alternating permutations

In this section, we aim to get the enumeration of 4123-avoiding up-down alternating permutations of odd and even length. We will show that 4123-avoiding up-down alternating permutations of length $2n+1$ are in one-to-one correspondence with standard Young tableaux of shape $(n+1, n, n-1)$. Moreover, for $n \geq 2$, there is a bijection between the set of 4123-avoiding up-down permutations of length $2n$ and the set of shifted standard Young tableaux of shape $(n+2, n, n-2)$. The following Lemma will be essential in establishing the bijections.

Lemma 3.1 *Let $\pi = \pi_1\pi_2 \dots \pi_n$ be a permutation in $\mathcal{DU}_n(4123)$ and a be a positive integer. If $a \leq \pi_1$, then $\pi' = a \to \pi$ is in $\mathcal{UD}_{n+1}(4123)$.*

Proof. Let $\pi' = \pi'_1\pi'_2 \dots \pi'_{n+1} = a \to \pi$. In order to prove $\pi' \in \mathcal{UD}_{n+1}(4123)$, it is sufficient to prove that there exists no subsequence $\pi'_1\pi'_i\pi'_j\pi'_k$ with $i < j < k$ in π'. Assume that $\pi'_1\pi'_i\pi'_j\pi'_k$ is a subsequence order-isomorphic to 4123. Since $\pi'_1 < \pi'_2$, we deduce that $\pi'_2\pi'_i\pi'_j\pi'_k$ is also a subsequence order-isomorphic to 4123, which implies that $\pi_1\pi_{i-1}\pi_{j-1}\pi_{k-1}$ is a subsequence order-isomorphic to 4123. This contradicts with the fact that π is a 4123-avoiding down-up alternating permutation. This completes the proof. ∎

Now we proceed to construct a map γ from the set $\mathcal{UD}_{2n+1}(4123)$ to the set of standard Young tableaux of shape $(n+1, n, n-1)$. Given a permutation $\pi = \pi_1\pi_2 \dots \pi_{2n+1} \in \mathcal{UD}_{2n+1}(4123)$, define $\pi' = \pi'_1\pi'_2 \dots \pi'_{2n}$ to be a permutation obtained from π by removing π_1 from π and deceasing each entry that is larger

than π_1 by one. Obviously, the permutation π' is in $\mathcal{DU}_{2n}(4123)$. By Theorem 2.8, the tableau $\bar{\phi}(\pi')$ is a standard Young tableau of shape (n, n, n) with the $(1, n)$-entry equal to $3n - \pi'_1$. Define $T = \gamma(\pi)$ to be a tableau obtained from $\bar{\phi}(\pi')$ by deleting the $(3, n)$-entry and inserting a $(1, n+1)$-entry equal to $(3n + 1 - \pi_1)$. Since $\pi_1 \leq \pi'_1$, the obtained tableau T is a standard Young tableau of shape $(n + 1, n, n - 1)$. Therefore, the map γ is well defined.

Theorem 3.2 *For $n \geq 1$, the map γ is a bijection between the set $\mathcal{UD}_{2n+1}(4123)$ and the set of standard Young tableaux of shape $(n + 1, n, n - 1)$.*

Proof. It is sufficient to construct the inverse mapping of γ to show that γ is a bijection. Given a standard Young tableau T of shape $(n+1, n, n-1)$, we wish to recover a permutation $\gamma^{-1}(T) \in \mathcal{UD}_{2n+1}(4123)$. Suppose that the $(1, n+1)$-entry and $(1, n)$-entry of T are equal to $3n + 1 - a$ and $3n - b$, respectively. Then we construct a permutation $\gamma^{-1}(T)$ as follows.

- Remove the $(1, n + 1)$-entry from the tableau T and decrease each entry that is larger than $3n + 1 - a$ by one;
- Insert a $(3, n)$-entry which is equal to $3n$. Denote by T' the obtained standard Young tableaux;
- Finally, set $\gamma^{-1}(T) = a \to \bar{\phi}^{-1}(T')$.

Note that T' is a standard Young tableau of shape (n, n, n) such that the $(1, n)$-entry equals $3n - b$. Let $\pi' = \bar{\phi}^{-1}(T') = \pi'_1\pi'_2 \ldots \pi'_{2n}$. By Theorem 2.8, we deduce that π' is a down-up alternating permutation in $\mathcal{DU}_{2n}(4123)$ with $\pi'_1 = b$. Since T is a standard Young tableau, we have $a \leq b$. By Lemma 3.1, the obtained permutation $\gamma^{-1}(T)$ is in $\mathcal{UD}_{2n+1}(4123)$. It is easy to verify that the construction of the map γ^{-1} reverses each step of the construction of the map γ. This completes the proof. ∎

Recall that there is a bijection between the set $\mathcal{UD}_{2n+1}(1234)$ and the set of standard Young tableaux of shape $(n + 1, n, n - 1)$ [5]. From Theorem 3.2, we deduce the following result.

Theorem 3.3 *For $n \geq 1$, we have*

$$|\mathcal{UD}_{2n+1}(4123)| = |\mathcal{UD}_{2n+1}(1234)|.$$

Example 3.4 *Consider a 4123-avoiding up-down alternating permutation $\pi = 4657132$. Let $\pi' = 546132$. The tableau $\gamma(\pi)$ is illustrated as*

1	2	4	5
3	6	9	
7	8		

For $n \geq 2$, given a permutation $\pi = \pi_1\pi_2\ldots\pi_{2n} \in \mathcal{UD}_{2n}(4123)$, let $\pi' = \pi'_1\pi'_2\ldots\pi'_{2n}$ be a permutation obtained from π by removing π_1 from π and deceasing each entry that is larger than π_1 by one. Obviously, the permutation π' is in $\mathcal{DU}_{2n-1}(4123)$. By Theorem 2.9, the tableau $\bar{\psi}(\pi')$ is a standard Young tableau of shape $(n+1, n, n-1)$ with the $(1, n+1)$-entry equal to $3n-\pi'_1$. Finally we obtain a tableau from $\bar{\psi}(\pi')$ by deleting the $(3, n-1)$-entry and inserting a $(1, n+1)$-entry equal to $(3n+1-\pi_1)$. Since $\pi_1 \leq \pi'_1$, the obtained tableau is a shifted standard Young tableau of shape $(n+2, n, n-2)$. Therefore we can deduce the following result by the similar arguments as in the proof of Theorem 3.2.

Theorem 3.5 *For $n \geq 2$, 4123-avoiding up-down alternating permutations of length $2n$ are in one-to-one correspondence with shifted standard Young tableaux of shape $(n+2, n, n-2)$.*

As in the case for standard Young tableaux, there is a simple hook length formula for shifted standard Young tableaux [2, 9]. By simple computation, we derive that the number of shifted standard Young tableaux of shape $(n+2, n, n-2)$ is equal to $\frac{2(3n)!}{n!(n+1)!(n+2)!}$. Recall that the number of 1234-avoiding up-down alternating permutations of length $2n$ is given by $\frac{2(3n)!}{n!(n+1)!(n+2)!}$. Hence, we obtain the following result.

Theorem 3.6 *For $n \geq 0$, we have*

$$|\mathcal{UD}_{2n}(4123)| = |\mathcal{UD}_{2n}(1234)| = \frac{2(3n)!}{n!(n+1)!(n+2)!}.$$

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